| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

| | |
|---|---|
| Surname: | MEMO |
| First Name(s): | |

Department of Mathematics and Applied Mathematics

MAM1000W

Test 2

Time : 18h00 – 20h00

3 May 2018

Full marks: 80

- This question paper consists of 16 pages (including this one). Pages are printed on both sides.
- Answer all questions in the spaces provided on this question paper.
- Use your *UCT Examination Answer Book* for rough work. The work in your *UCT Examination Answer Book* will not be marked.
- Calculators are not allowed. When your answer contains constants such as $\pi, e$ or $\sqrt{2}$, leave them in that form. Don't simplify your answers.
- Be careful to provide answers that we can read and make sense of at all times. Work that is poorly presented will be penalized.

DO NOT WRITE BELOW THIS LINE

| | | | | | Subtotals |
|---|---|---|---|---|---|
| A | | correct | | incorrect | |
| | 4 times | | minus | | /24 |
| B | | | B1 | B2 | |
| | | | /12 | /4 | /16 |

1

| | C1 | C2 | C3 | C4 | C5 | C6 | C7 | C8 | |
|---|---|---|---|---|---|---|---|---|---|
| C | /4 | /4 | /4 | /4 | /5 | /8 | /5 | /6 | /40 |

| | |
|---|---|
| Total | /80 |
| Check 1 | initials |
| Check 2 | initials |

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

## SECTION A: MULTIPLE CHOICE QUESTIONS [24 marks]

Marking: Correct answer = 4, no answer = 0, wrong answer = -1

| Question: | A1 | A2 | A3 | A4 | A5 | A6 |
|---|---|---|---|---|---|---|
| Answer: | A | D | B | B | E | B |

**Question A1.** [4 points] The equation for the tangent to the curve $y = x^2 + \sin x$ at the point where $x = \frac{\pi}{2}$ is

$\boxed{\text{A: } y = 1 - \frac{\pi^2}{4} + \pi x}$ B: $y = (1 - \frac{\pi^2}{4})x + \pi$ C: $y = 1 + \frac{\pi^2}{4} + \pi x$ D: $y = (2 + \frac{\pi^2}{4})x + \pi$

E: $y = (2 + \frac{\pi^2}{4})x + \pi$

**Question A2.** [4 points] Suppose $f$ and $g$ are differentiable functions for which you know the following:

| $x$ | $f(x)$ | $f'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|---|
| 1 | 3 | 4 | $-1$ | 3 |
| 2 | 1 | $-3$ | 6 | 2 |

Then

$$\frac{d}{dx} g\left(\sqrt{f(x)}\right)\bigg|_{x=2}$$

is equal to

A: $\frac{9}{2}$ B: $\frac{3}{2}$ C: $-\frac{3}{2}$ $\boxed{\text{D: } -\frac{9}{2}}$ E: -9

**Question A3.** [4 points] The value of $\lim_{x \to \infty}(\sqrt{x^2 + ax} - \sqrt{x^2 + bx})$ is

A: $a - b$ $\boxed{\text{B: } \frac{a-b}{2}}$ C: $\frac{a+b}{2}$ D: $a + b$ E: Undefined

**Question A4.** [4 points] $\tan(\arcsin x)$ is equal to is

A: $\frac{x}{\sqrt{1+x^2}}$ $\boxed{\text{B: } \frac{x}{\sqrt{1-x^2}}}$ A: $\frac{1}{\sqrt{1+x^2}}$ B: $\frac{1}{\sqrt{1-x^2}}$ E: $\frac{1}{\sqrt{x^2-1}}$

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

Question A5. [4 points] For the function $f(x) = \frac{1}{x+4}$, which of the following statements are true:

- S1: $\lim_{x\to -4^+} f(x)$ and $\lim_{x\to -4^+} f(x)$ are real numbers but are not equal.
- S2: $\lim_{x\to -4} f(x)$ does not exist.
- S3: $f(-4)$ and $\lim_{x\to -4} f(x)$ exist but are not equal.
- S4: $f(-4)$ is undefined.

A: S1, S2 and S4 are true

B: S1, S3 are S4 are true

C: S1 and S4 are true

D: S2 and S3 are true

E: S2 and S4 are true

Question A6. [4 points] Let $f(x) = \tan(x)$, $g(x) = |x-1|$ and $h(x) = \frac{x^2+3x}{2+4x}$. Which of them are differentiable everywhere they are defined?

A: $f$ and $g$

B: $f$ and $h$

C: $g$ and $h$

D: $f$

E: None is differentiable everywhere it is defined.

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

SECTION B: SHORT ANSWER QUESTIONS [16 marks]

Write the answers to the questions in the blocks provided. Do not show your working.

Question B1. [12 points] Calculate the following derivatives:

a. $\frac{d}{dx}\left(\frac{\tan x}{x^3}\right)$ (3)

$$\frac{x\sec^2 x - 3\tan x}{x^4}$$

b. $\frac{d}{dx} x\sec x$ (3)

$$\sec x(1 + x\tan x)$$

c. $\frac{d}{dx}\sqrt[3]{x}(x^2-1)$ (3)

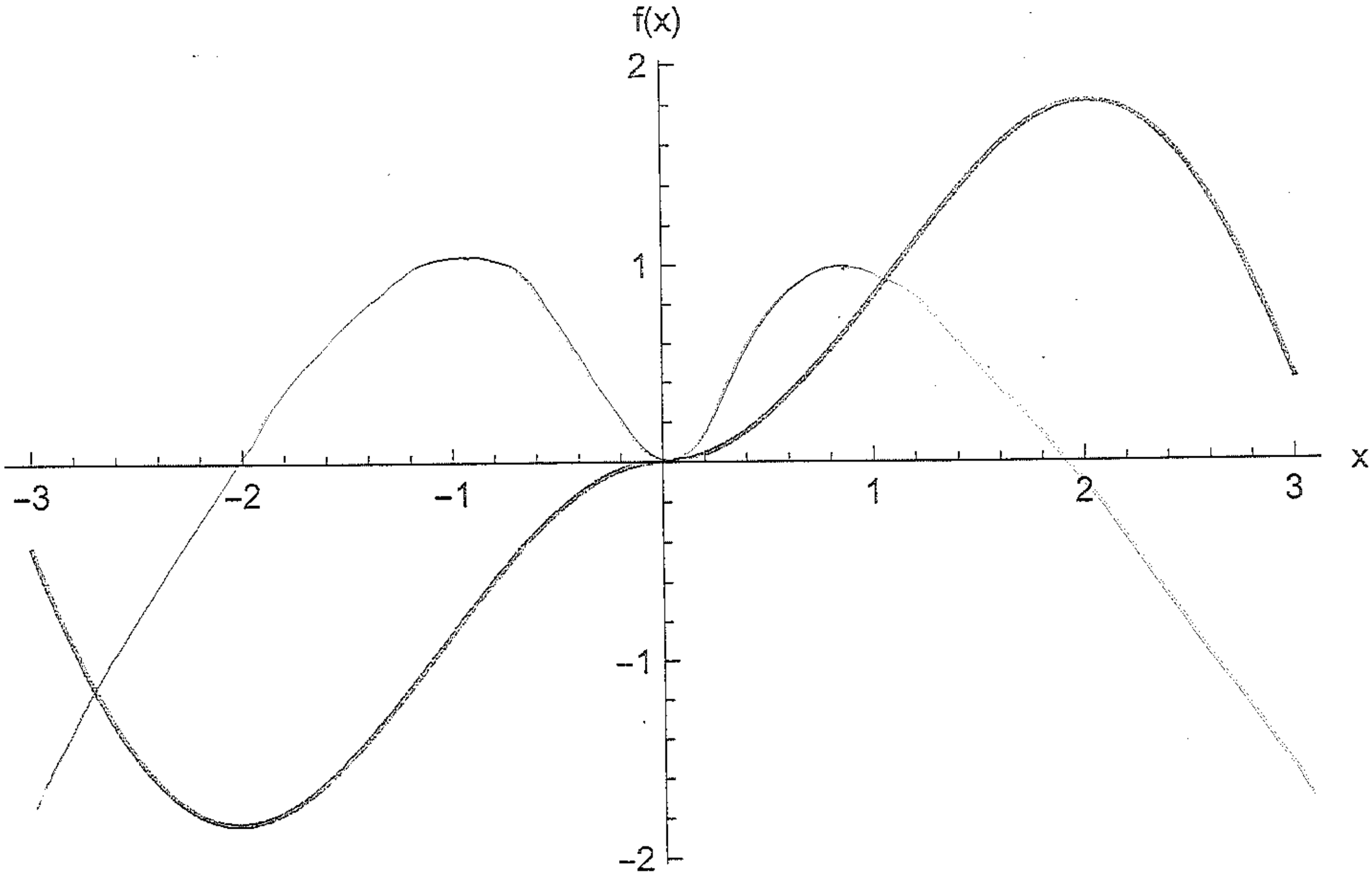
$$\frac{7x^2-1}{3x^{2/3}}$$

d. $\frac{d}{dy}\frac{\sin y\cos y}{y^2}$ (3)

$$\frac{y\cos(2y) - \sin(2y)}{y^3}$$

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

Question B2. [4 points] On the following graph of $y = f(x)$, sketch the graph of $y = f'(x)$.



| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

## SECTION C: LONG ANSWER QUESTIONS [40 marks]

Write the answers to the questions in the spaces provided.
**Show all working!**

**Question C1.** [**4 points**] If a function $f$ is given by $f(x) = \dfrac{3}{x-4}$, find any points of intersection of the graph of $f$ with the graph of the inverse function $f^{-1}$.

if $f(x) = \dfrac{3}{x-4}$

$f^{-1}(x)$ is found from solving $y = \dfrac{3}{x-4}$

$$x - 4 = \frac{3}{y}$$

$$x = \frac{3}{y} + 4$$

$\therefore f^{-1}(x) = \dfrac{3}{x} + 4$

points of intersection when $\dfrac{3}{x-4} = \dfrac{3}{x} + 4$

$$3x = (x-4)(3x+4)$$

$$0 = x^2 - 4x - 3$$

$$x = \frac{-4 \pm \sqrt{(-4)^2 + 4\cdot 3}}{2}$$

$$= 2 \pm \sqrt{7}$$

at $x = 2 \pm \sqrt{7}$, $y = \dfrac{3}{2 \pm \sqrt{7} - 4}$. The points of intersection are

$\left(2+\sqrt{7}, \dfrac{3}{\sqrt{7}-2}\right)$ and $\left(2-\sqrt{7}, \dfrac{3}{-2-\sqrt{7}}\right)$

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

Question C2. [4 points] Evaluate the limit

$$\lim_{x\to 0} \frac{\sqrt{x^2+16}-4}{x^2}.$$

(you may not use L'Hospital's rule)

$$\lim_{x\to 0} \frac{\sqrt{x^2+16}-4}{x^2} = \lim_{x\to 0} \frac{\sqrt{x^2+16}-4}{x^2} \frac{(\sqrt{x^2+16}+4)}{(\sqrt{x^2+16}+4)}$$

$$= \lim_{x\to 0} \frac{x^2+16-16}{x^2(\sqrt{x^2+16}+4)}$$

$$= \lim_{x\to 0} \frac{x^2}{x^2(\sqrt{x^2+16}+4)} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+16}+4}$$

$$= \frac{1}{4+4} = \frac{1}{8}$$

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

Question C3. [4 points] Prove that there is a real number $x$ such that $3^x + \sin(x) = 4$.

if there exists $x \in \mathbb{R}$ such that $3^x + \sin x = 4$

then this $x$ solves $3^x + \sin x - 4 = 0$

$3^x, \sin x, -4$ are all continuous so

$3^x + \sin x - 4$ is continuous.

~~If we can~~ for $x = 0$, $3^x + \sin x - 4$

$= 3^0 + \sin 0 - 4$

$= 1 + 1 - 4$

$= -2$

for $x = 2$, $3^x + \sin x - 4$

$= 9 - 4 + \sin 2$

$= 5 + \sin 2 > 0$

so by the IVT there exists a value in $(0,2)$

such that $3^x + \sin x = 4$

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

---

Question C4. [4 points] Use the squeeze theorem and the limit laws to find the value of

$$\lim_{x\to 0}\left(x^2 e^{\sin\left(\frac{1}{x}\right)} + 4\right).$$

$$\lim_{x\to 0}\left(x^2 e^{\sin\left(\frac{1}{x}\right)} + 4\right) \cancel{= \lim_{x\to 0} x^2 e^{\sin\left(\frac{1}{x}\right)} + \lim_{x\to 0} 4}$$

$$e^{-1} \leq e^{\sin\left(\frac{1}{x}\right)} \leq e^{1}$$

$$x^2 e^{-1} \leq x^2 e^{\sin\left(\frac{1}{x}\right)} < x^2 e$$

$$\lim_{x\to 0} x^2 e^{-1} \leq \lim_{x\to 0} x^2 e^{\sin\left(\frac{1}{x}\right)} \leq \lim_{x\to 0} x^2 e$$

$$0 \leq \lim_{x\to 0} x^2 e^{\sin\left(\frac{1}{x}\right)} \leq 0$$

$\therefore$ by the squeeze theorem

$$\lim_{x\to 0} x^2 e^{\sin\left(\frac{1}{x}\right)} = 0$$

$$\lim_{x\to 0} x^2 e^{\sin\left(\frac{1}{x}\right)} + 4 = \lim_{x\to 0} x^2 e^{\sin\left(\frac{1}{x}\right)} + \lim_{x\to 0} 4$$

$$= 0 + 4 = 4$$

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

Question C5. [5 points] Prove by contradiction that $\sqrt[3]{2}$ is irrational.

Assume that $\sqrt[3]{2}$ is rational, ie.

$\sqrt[3]{2} = \frac{a}{b}$, $a, b \in \mathbb{Z}$, a & b have no common factors and $b \neq 0$

$2 = \frac{a^3}{b^3} \Rightarrow a^3 = 2b^3$

$\therefore$ $a^3$ is even, $a^3$ is divisible by 2

$a^3$ has a unique prime decomposition, so if $a^3$ has a factor of 2, then so does a. $\therefore$ $a = 2k$, $k \in \mathbb{Z}$

$(2k)^3 = 2b^3 \Rightarrow 4k^3 = b^3$

$\therefore$ $b^3$ is divisible by 4. $\therefore$ it is divisible by 2. As above, this implies that b is divisible by 2.

$\therefore$ a & b have a common factor $\Rightarrow$ contradiction. Therefore $\sqrt[3]{2}$ is irrational

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

Question C6. [8 points] Define

$$f(x) = \begin{cases} x^3 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

a. Show that $f(x)$ is continuous at $x = 0$. (4)

2 ways : 1) look at $\lim_{x \to 0} x^2 \sin(\frac{1}{x})$

$$-1 \le \sin(\tfrac{1}{x}) \le 1$$

$$-x^2 \le x^2 \sin(\tfrac{1}{x}) \le x^2$$

$$\lim_{x\to 0}(-x^2) \le \lim_{x\to 0} x^2 \sin(\tfrac{1}{x}) \le \lim_{x\to 0}(x^2)$$

$$0 \le \lim_{x\to 0} x^2 \sin(\tfrac{1}{x}) \le 0$$

so $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$ by the squeeze theorem.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[x^3 \sin(\tfrac{1}{x})\right] = \lim_{x\to 0} x \lim_{x\to 0} x^2 \sin\tfrac{1}{x}$$

$$= 0.0 = 0 = f(0)$$

$\therefore$ continuous.

2) Can also apply the squeeze theorem to $x^3 \sin(\frac{1}{x})$ but you have to treat $\lim_{x\to 0^+}$ and $\lim_{x\to 0^-}$ separately.

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

b. Use first principles to decide whether $f(x)$ is differentiable at $x = 0$. (4)

$$\frac{d}{dx} f(x)\Big|_{x=0} = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)-0}{h}$$

$$= \lim_{h\to 0} \frac{h^3 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} h^2 \sin\left(\frac{1}{h}\right)$$

$= 0$ by the intermediate step in the previous question.

Since $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0$ exists, $f(x)$ is differentiable at $x=0$

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

Question C7. [5 points] Use proof by induction to show that for all integers $n \geq 2$

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\dots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}$$

~~The~~ base case is $n=2$, check $\Rightarrow$

$$\left(1-\frac{1}{4}\right)=\frac{2+1}{2\times 2}=\frac{3}{4} \checkmark$$ base case holds.

Assume that it holds true for some $n=k$ $(k \geq 2)$

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\dots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$$

Then for $k+1$, the L.H.S is

By the inductive hypothesis

$$\underbrace{\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\dots\left(1-\frac{1}{k^2}\right)}\left(1-\frac{1}{(k+1)^2}\right)=\left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$\left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)=\left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$=\frac{k^2+2k+1-1}{(2k)(k+1)}=\frac{k(k+2)}{2k(k+1)}=\frac{k+2}{2(k+1)}$$

$$=\frac{(k+1)+1}{2(k+1)}$$

~~this~~ which $=\frac{n+1}{2n}$ for $n=k+1$. $\therefore$ by mathematical induction, the statement holds for all integers $n \geq 2$

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

Blank Page

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

Question C8. [6 points] Prove, from first principles, that if $f(x)$ is a differentiable function, then for $f(x) \neq 0$, so is the function $g$ defined by $g(x) = \frac{x}{f(x)}$, and that $g'(x) = \dfrac{1}{f(x)} - \dfrac{xf'(x)}{f(x)^2}$.

In answering this question, you may not use, quote or prove the general rule for the derivative of a product or quotient of functions or the composition of functions.

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{\frac{x+h}{f(x+h)} - \frac{x}{f(x)}}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)f(x) - x f(x+h)}{h f(x) f(x+h)} = \lim_{h \to 0} \frac{x f(x) + h f(x) - x f(x+h)}{h f(x) f(x+h)}$$

$$= \lim_{h \to 0} \frac{h f(x) - x [f(x+h) - f(x)]}{h f(x) f(x+h)}$$

$$= \lim_{h \to 0} \frac{1}{f(x+h)} \left[ 1 - \frac{x}{f(x)} \cdot \frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h \to 0} \frac{1}{f(x+h)} \cdot \lim_{h \to 0} \left[ 1 - \frac{x}{f(x)} \frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h \to 0} \frac{1}{f(x+h)} \cdot \left[ \lim_{h \to 0} 1 - \lim_{h \to 0} \frac{x}{f(x)} \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right]$$

$$= \frac{1}{f(x)} \left[ 1 - \frac{x}{f(x)} f'(x) \right] = \frac{1}{f(x)} - \frac{x f'(x)}{f(x)^2}$$

| Student Number: | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|

Blank Page